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LATIN SQUARES

WITH DIRECTIONS FOR MAKING A RANDOM SELECTION

By K. Brecheen, In Charge Statistical Division,
California Forest and Range Experiment Station
and C. B. Hess, WPA Research Associate Statistician

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Agricultural experiments make use of techniques designed to minimize the effect on comparisons, of soil heterogeneity. The Latin Square is one of the patterns used in assigning different experimental procedures to an area; from the Latin square and the correct statistical treatment of resulting data comparisons of the treatments and their statistical significance can be determined to a degree of accuracy impossible by previously employed arrangements.

The theory of experimental design created by R. A. Fisher is completely described in his latest book[1] and mentioned in various other sources. From these publications the basic plan of an experiment, the theory underlying it, and the appropriate method of analysis of the resulting data can be completely obtained. The fact is pointed out that a simultaneous comparison of a number of treatments can be set up according to several different arrangements, among which is the Latin square, an arrangement especially suitable to a large variety of agricultural problems.

A Latin square of dimensions nxn is used to test out n treatments, each one being repeated n times. Characteristic of the Latin square is the requirement that each treatment occur only once in each row, and once in each column. Since each row contains each treatment, the rows can be compared, and systematic differences between rows can be measured and eliminated from the estimate of experimental error. This feature, or, as Fisher states,[1] (pages 78-79) "the knowledge that there is less variation within certain aggregates of these" (plots of land, experimental animals, etc.) "than there is among different individuals belonging to different aggregates", makes possible the increased precision resulting from use of the Latin square and other arrangements.

1/ Fisher, R. A. The design of experiments. Oliver and Boyd, London. 1935.

For the statistical treatments mentioned above, experimental variables must be located at random in the Latin square. Since varying the locations of these variables results in a different Latin square, it is apparent that one Latin square is being chosen out of many of the same dimensions. A completely random selection should give each Latin square of those dimensions an equal chance of being selected. Practical attempts to select a single nxn Latin square make obvious the need for this article: the total number of Latin squares of various sizes is so great, and their study and enumeration are scattered over so many sources, that selecting even a single one at random may require considerable research for an understanding of their general characteristics. The purpose of this article is to summarize simply these characteristics; to list the simplest methods of selecting at random Latin squares of order 2x2, 3x3, 4x4, 5x5, 6x6, and higher; and to enumerate as far as practicable the various types known.

Characteristics of Latin Squares

A Latin square has been defined as a square of dimensions nxn, representing n replications of n different treatments, such that each treatment occurs only once in each row and once in each column. If each treatment is represented by a letter, examples are the 2x2 and 3x3 Latin squares below.

(1)	(2)
A B	B C A
B A	C A B
	A B C

There are
2 different 2x2 Latin squares possible,
12 different 3x3 Latin squares possible,
576 different 4x4 Latin squares possible,
161,280 different 5x5 Latin squares possible,
812,851,200 different 6x6 Latin squares possible,

and an undetermined number of Latin squares of dimensions 7x7 and over.

Obviously there are too many possibilities to list for a square of dimensions 6x6 or even 5x5. The following characteristics of Latin squares aid in summarizing all possibilities.

1. A Latin square with the first row and first column in alphabetical order A B C D E F .... is called a reduced Latin square. The above 3x3 square (2), can be transformed into a reduced square by rearranging columns so that the last one precedes the other two; the order of columns becomes 3, 1, 2.

(3)

A B C
B C A
C A B

Each reduced square may be transformed into n! (n-1)! different Latin squares by rearranging the rows in n! ways and the columns in (n-1)! ways, or vice versa. (The above reduced square can be transformed into 3! (2)! different Latin squares, by rearranging rows and columns.)

2. The process of rearrangement, or permutation, of rows, columns, or letters of a Latin square among themselves is called transformation, and the total of all Latin squares resulting from application to a given square of such a procedure, is called a transformation set. Every member of a transformation set can be obtained from every other member of the transformation set by such permutations; but no member of a second transformation set can be obtained from any member of the first set by permuting rows, columns, and/or letters. The first 3x3 Latin square (2) and the 3x3 square in reduced form (3) are members of the same transformation set.

3. The number of transformation sets possible varies with the dimensions of the square concerned.

2x2 Latin squares have 1 transformation set,
3x3 Latin squares have 1 transformation set,
4x4 Latin squares have 2 transformation sets,
5x5 Latin squares have 2 transformation sets,
and 6x6 Latin squares have 22 transformation sets.

Each transformation set may include a different number of reduced squares. One of the two sets of dimensions 4x4, includes three reduced squares and the other set only one. And of the two transformation sets found for dimensions 5x5, one set includes 6 reduced squares while the other includes 50.

Although the number of transformation sets varies with the dimensions of the Latin square involved, and although the number of reduced squares included varies with the particular transformation set they represent, yet each reduced square gives rise to n! (n-1)! different Latin squares of dimensions nxn, by permuting rows and columns.

From these facts it is clear that one can obtain mechanically all the possible Latin squares in several ways; from a complete list of the reduced squares of a certain size, one can get all of that size by permuting (changing the order of) only rows and columns; or from a list of representatives from each transformation set of a given size, one can obtain all possible Latin squares of that size by permuting rows, columns, and letters. The method here recommended varies with the size of the square considered. For convenience to the reader, each size is treated independently, not only in the description of methods of selection but also in specific examples in the Appendix. A summary of information relating to squares of various dimensions is given below in Table 1 together with an outline of the techniques suggested for randomizing one or more selections.2/

2/ The enumerations of Latin squares up to dimensions 6x6 were worked out by authors and checked against the following sources:

Cayley, A. On Latin squares. The Messenger of Mathematics. 19: 135-137, 1890.
Fisher, R. A. The design of experiments. Oliver and Boyd, London. 1935.
Fisher, R. A. and J. Wishart. The arrangement of field experiments and the statistical reduction of the results. Imp. Bureau of Soil Science, Tech. Com. No. 10, London, 1930.
Fisher, R. A. Statistical methods for research workers, 6th Ed. Oliver and Boyd, London.

Table 1. - Summary of characteristics of Latin Squares and techniques for randomizing selections.

Size of square	Transformation set no.	No. reduced squares in set	Total No. squares possible	Technique of randomizing a selection
2x2	#1	1	2	From list of total possible, select one.
3x3	#1	1	12	From list of total possible, select one.
4x4	#1	1	144	From list of reduced squares, select one and permute rows and columns.
	#2	3	432	
Total	2	4	576	
5x5	#1	50	144,000	From list of reduced squares, select one and permute rows and columns.
	#2	6	17,280	
Total	2	56	161,280	
6x6	#1	1080		From list of samples of transformation sets, select one and permute rows, columns and letters. (Method of selecting explained on page 10.)
	#2	1080		
	#3	1080		
	#4	1080		
	#5	1080		
	#6	540		
	#7	540		
	#8	540		
	#9	540		
	#10	360		
	#11	360		
	#12	360		
	#13	180		
	#14	120		
	#15	120		
	#16	120		
	#17	60		
	#18	40		
	#19	36		
	#20	36		
	#21	36		
	#22	20		
Total	22	9408	812,851,200	
7x7	?	?	?	Work out by hand.

Each reduced square has 6! 5! or 86,400 variations.

Directions for Selecting a Latin Square at Random

It is assumed that the experimenter has already assigned letters to the treatments he is testing. One often finds that the variables have a logical order such that the procedure of experimentation and the interpretation of results are facilitated by keeping their relationship obvious.

Assuming that letters A, B, C, D, E,... have been assigned to a corresponding number of treatments respectively and that this assignment remains constant throughout all Latin squares selected, the following methods of randomization are suggested:

2x2 Latin Squares

Select at random one of the two possible 2x2 Latin squares below.

1.	2.
A B	B A
B A	A B

3x3 Latin Squares

Select at random one of the 12 possible 3x3 Latin squares below.

1.	2.	3.	4.	5.	6.
A B C	C A B	B A C	A B C	C A B	B A C
B C A	A B C	C B A	C A B	B C A	A C B
C A B	B C A	A C B	B C A	A B C	C B A

7.	8.	9.	10.	11.	12.
B C A	A C B	C B A	B C A	A C B	C B A
C A B	B A C	A C B	A B C	C B A	B A C
A B C	C B A	B A C	C A B	B A C	A C B

4x4 Latin Squares

To obtain a random arrangement in a 4x4 Latin square:

1. Select at random one of the four reduced squares below.

2. Select at random the order of rows, and rewrite in the new order.

3. Select at random the order of the last three columns, leaving the first column unchanged; rewrite in the new order. This is a random choice from the 576 possible 4x4 Latin squares.

The four possible reduced squares are:[3/]

1.	2.	3.	4.
A B C D	A B C D	A B C D	A B C D
B A D C	B A D C	B D A C	B C D A
C D A B	C D B A	C A D B	C D A B
D C B A	D C A B	D C B A	D A B C

5x5 Latin Squares

To obtain a random arrangement in a 5x5 Latin square:

1. Select at random one of the 56 reduced squares below.

2. Select at random the order of rows, and rewrite in the new order.

3. Select at random the order of the last four columns, leaving the first column unchanged: rewrite in the new order. This is a random choice from the 161,280 possible 5x5 Latin squares.

3/ The first reduced square belongs to one transformation set and the last three belong to the other.

5x5 Latin Squares[4]

1.	2.	3.	4.	5.
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B A E C D	B A D E C	B A E C D	B A D E C	B D A E C
C D A E B	C E A B D	C D B E A	C E B A D	C E D B A
D E B A C	D C E A B	D E A B C	D E B C A	D C E A B
E C D B A	E D B A C	E C D A B	E C A B D	E A B C D

6.	7.	8.	9.	10.
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B E A D C	B C E A D	B E D A C	B C D E A	B D E C A
C D E A B	C D A E B	C A E B D	C E A B D	C A D E B
D A B E C	D E B C A	D C A E B	D A E C B	D E B A C
E C D B A	E A D B C	E D B C A	E D B A C	E C A B D

11.	12.	13.	14.	15.
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B C A E D	B C A E D	B D E A C	B D E A C	B E D C A
C D E A B	C E D B A	C A D E B	C E D B A	C A E B D
D E B C A	D A E C B	D E B C A	D C A E B	D C A E B
E A D B C	E D B A C	E C A L D	E A B C D	E D B A C

16.	17.	18.	19.	20.
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B E D C A	B D A E C	B D A E C	B D A E C	B E A C D
C D E A B	C E B A D	C E D A B	C E D B A	C D B E A
D A B E C	D C E B A	D C E B A	D A E C B	D A E B C
E C A B D	E A D C B	E A B C D	E C B A D	E C D A B

21.	22.	23.	24.	25.
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B E A C D	B C E A D	B C E A D	B C E A D	B C E A D
C D E B A	C D E A B	C A D E B	C D B E A	C D B E A
D A B E C	D C B E A	D E B C A	D E A B C	D E A C B
E C D A B	E A D B C	E D A B C	E A D C B	E A D B C

26.	27.	28.	29.	30.
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B E D A C	B E D A C	B E D A C	B C D E A	B C D E A
C A B E D	C A E B D	C D E B A	C A E B D	C E B A D
D E A C B	D C B E A	D C A E B	D E A C B	D A E B C
E A D B C	E D A C B	E A B C D	E D B A C	E D A C B

4/ The first 50 reduced squares belong to one transformation set, and the last 6 to the other.

5x5 Latin Squares (Cont'd)

31.

A B C D E
B C D E A
C E B A D
D A E C B
E D A B C

32.

A B C D E
B D E C A
C A B E D
D E A B C
E C D A B

33.

A B C D E
B D E C A
C A D E B
D E A B C
E C B A D

34.

A B C D E
B D E C A
C E D A B
D A B E C
E C A B D

35.

A B C D E
D A E C B
C E D A B
D C B E A
E D A B C

36.

A B C D E
B A E C D
C E D B A
D C A E B
E D B A C

37.

A B C D E
B A D E C
C D E B A
D E A C B
E C B A D

38.

A B C D E
B A D E C
C D E A B
D E B C A
E C A B D

39.

A B C D E
B C A E D
C D E B A
D E B A C
E A D C B

40.

A B C D E
B C A E D
C E D A B
D A E B C
E D B C A

41.

A B C D E
B D E A C
C A B E D
D E A C B
E C D B A

42.

A B C D E
B D E A C
C E A B D
D C B E A
E A D C B

43.

A B C D E
B E D C A
C A B E D
D C E A B
E D A B C

44.

A B C D E
B E D C A
C D A E B
D A E B C
E C B A D

45.

A B C D E
B D A E C
C E B A D
D A E C B
E C D B A

46.

A B C D E
B E A C D
C D B E A
D C E A B
E A D B C

47.

A B C D E
B C E A D
C A D E B
D E A B C
E D B C A

48.

A B C D E
B E D C A
C A B E D
D C E A B
E D A B C

49.

A B C D E
B C D E A
C A E B D
D E B A C
E D A C B

50.

A B C D E
B D E C A
C E A B D
D A B E C
E C D A B

51.

A B C D E
B D A E C
C A E B D
D E B C A
E C D A B

52.

A B C D E
B E A C D
C A D E B
D C E B A
E D B A C

53.

A B C D E
B C E A D
C E D B A
D A B E C
E D A C B

54.

A B C D E
B E D A C
C D B E A
D A E C B
E C A B D

55.

A B C D E
B C D E A
C D E A B
D E A B C
E A B C D

56.

A B C D E
B D E C A
C E B A D
D C A E B
E A D B C

6x6 Latin Squares

Samples of the 22 transformation sets for 6x6 Latin squares are listed. Theoretically, since each reduced square can give rise without duplication to the same number of different squares, it would be adequate to select one reduced square at random and then permute rows and columns.

The 9408 reduced squares of dimensions 6x6 are too numerous to list here however. The modification of selecting at random one transformation set is adopted since all the reduced squares in that set may be obtained from one member. But since there are not as many reduced squares in some sets as in others, it is necessary to weight the transformation sets so that the chance of each one's being chosen is proportional to the total number of reduced squares it includes. The key numbers 1-9408 shown above the sets listed are used for this purpose.

To obtain a random arrangement in a 6x6 Latin square:

1. Select at random one of the numbers 1-9408. Locate the transformation set in which this theoretical reduced square occurs and record the sample listed.

2. Select at random the order of rows, and rewrite in the new order.

3. Select at random the order of columns, and rewrite in the new order.

4. Select at random the order of letters, and rewrite in the new order.

The resulting 6x6 Latin square is a random selection from the known 812,851,200 possible.

6x6 Latin Squares[5/]

Samples of the 22 transformation sets known for 6x6 Latin Squares,
One reduced square from each set.

0001-1080

A B C D E F
B C F A D E
C F B E A D
D E A B F C
E A D F C B
F D E C B A

1081-2160

A B C D E F
B C F E A D
C F B A D E
D A E B F C
E D A F C B
F E D C B A

2161-3240

A B C D E F
B C F E A D
C F B A D E
D E A B F C
E A D F C B
F D E C B A

3241-4320

A B C D E F
B A F E C D
C F B A D E
D C E B F A
E D A F B C
F E D C A B

4321-5400

A B C D E F
B A E F C D
C F B A D E
D E A B F C
E D F C B A
F C D E A B

5401-5940

A B C D E F
B A E C F D
C F B A D E
D E F B C A
E D A F B C
F C D E A B

5941-6480

A B C D E F
B A F E D C
C E B F A D
D C A B F E
E F D C B A
F D E A C B

6481-7020

A B C D E F
B A F E C D
C F B A D E
D E A B F C
E C D F B A
F D E C A B

7021-7560

A B C D E F
B C D E F A
C E A F B D
D F B A C E
E D F B A C
F A E C D B

7561-7920

A B C D E F
B A E F C D
C F A E D B
D C D A F E
E D F C B A
F E D B A C

7921-8280

A B C D E F
B A F C D E
C E A B F D
D F E A C B
E C D F B A
F D B E A C

8281-8640

A B C D E F
B A E F C D
C F A B D E
D E B A F C
E D F C B A
F C D E A B

8641-8820

A B C D E F
B C F A D E
C F B E A D
D A E B F C
E D A F C B
F E D C B A

8821-8940

A B C D E F
B C A F D E
C A B E F D
D F E B A C
E D F C B A
F E D A C B

8941-9060

A B C D E F
B C A F D E
C A D E F B
D F E B C A
E D F A B C
F E D C A B

9061-9180

A B C D E F
B C A E F D
C A B F D E
D E F B A C
E F D A C B
F D E C B A

9181-9240

A B C D E F
B C D E F A
C D E F A B
D E F A B C
E F A B C D
F A B C D E

9241-9280

A B C D E F
B C A E F D
C A B F D E
D F E B A C
E D F C B A
F E D A C B

9281-9316

A B C D E F
B A F E D C
C D A B F E
D F E A C B
E C B F A D
F E D C B A

9317-9352

A B C D E F
B A D F C E
C F A E B D
D E B A F C
E D F C A B
F C E B D A

9353-9388

A E C D E F
B A E C F D
C E A F D B
D C F A B E
E F D B A C
F D B E C A

9389-9408

A B C D E F
B C A F D E
C A B E F D
D E F A B C
E F D C A B
F D E B C A

5/ Reproduced from: Fisher, R. A. and Yates. The 6x6 Latin Squares. Proceedings Camb. Phil. Soc. 30: 492-507. Oct. 30, 1934.

Latin Squares Larger than 6x6

Although 6x6 Latin squares seem the most usable size in agricultural experimentation, occasionally seven or more treatments occur to be compared simultaneously. With so many variables the possible number of arrangements increases at an amazing rate, and as yet no enumeration has been accomplished for Latin squares with more than six units on a side.

Randomization for these larger squares must be done by locating individual letters at random in the square. The usual method is to start at the upper left corner and determine the order of letters in row 1; then go to row 2, and so down. Difficulties arise in lower rows when, for instance, the only space left for a certain letter cannot be used because that letter is already represented in its column.

The method here suggested consists of selecting one of the n letters and determining its position by chance in each row beginning at the top; then a second letter is selected and completely located by chance in each row beginning again at the top. The possibility of not having space for a given letter in any row is avoided by systematically changing, when the need develops, the order of rows filled in so that those rows which have the least spaces usable, are finished first.

To select at random one square of dimensions larger than 6x6, specific directions are as follows:

1. Determine locations of the first letter so that it appears once in each row and once in each column. Start with row 1 and work down, determining the column at random.

2. Continue until as many letters have been filled in as can satisfy the above requirements. When a letter is found for which the above requirement cannot be satisfied without "backing up" and replacing one of the random locations with another, this procedure changes. Remove all random locations of this letter and use the method described under 3.

3. Record on one side of the square the number of choices possible for the letter under consideration, for each row. This number will be the same for all rows. Determine by random selection which space the letter will occupy in the first row, and change the list of possibilities at the side of the square to correspond with the changed number of possibilities remaining. Several lines will have the original number of choices and a few will have one less than the original number.

Now fill in the top row that has least vacant usable spaces and change the number of possibilities as they are affected by this choice. Continue in this way, systematically locating the letter first in rows with the fewest possible openings for the given letter.

4. Repeat above operations for each of the remaining letters, in their order. The resulting nxn Latin square is a random choice from the unknown total of nxn different squares possible.

APPENDIX

The following examples suggest the use of a table of random numbers. Such a table is simple to use but is only one of a number of devices which might be employed in making a random choice.

Example 1: Random choice of one or a series of 3x3 Latin squares, assuming that the letters A, B, C have already been assigned to variables.

34
07
77

The series of numbers to the left is in random order and will indicate the use that can be made of random sampling numbers.

45
17

Selecting at random one of the twelve possible 3x3 Latin squares, we pick up one of the series, say 34, divide it by 12, and get the remainder 10. Latin square Number 10 is our random choice. The 3x3 Latin square numbered 10 in the list on page 6 is

B C A
A B C
C A B

Example 2: Random choice of one or a series of 4x4 Latin squares, assuming that the letters A, B, C, D have already been assigned to the variables.

66
57
25
92

1. Select at random one of the 4 reduced squares using the random numbers at the left; the remainder when 66 is divided by 4 is 2. The reduced square numbered (2) from the list on page 7 is

73

A B C D
B A D C
C D B A
D C A B

2. To select at random the order in which these rows should be arranged: if we use the last digit of the random numbers 57, 25 etc., we find 7 ÷ 4 has a remainder of 3; 5 ÷ 4 has a remainder of 1; 2 ÷ 4 gives the remainder 2; and the last of the four selections must be number 4. The order of the rows is therefore: 3, 1, 2, 4 and the final square is

C D B A
A B C D
B A D C
D C A B

Example 3: Random choice of one or a series of 5x5 Latin squares, assuming that the letters A, B, C, D, E have already been assigned to the variables.

720
591
788
382
220
543
338
357
317

1. We select at random one of the 56 reduced squares; using the random numbers at the left, the remainder when 720 is divided by 56 is 48. The 48th reduced square listed on page 9 is

A B C D E
B E D C A
C A B E D
D C E A B
E D A B C

2. To select at random the order in which these rows should be arranged; if we use the last two digits of the random numbers 591, 788 etc., we find 91 ÷ 5 has a remainder of 1; 88 ÷ 5 has a remainder of 3; 82 ÷ 5 has a remainder of 2; 20 ÷ 5 gives the remainder 5; and the last of the five must be number 4. The order of rows is thus 1, 3, 2, 5, 4, and the square rearranged for rows is

A B C D E
C A B E D
B E D C A
E D A B C
D C E A B

3. Selecting at random the order of the last 4 columns, 43 divided by 4 gives the remainder 3; 38 gives the remainder 2; 57 gives 3, which is discarded; 17 gives the remainder 1; and the last column must be 4. The order of columns is A 3 2 1 4 or A D C B E. The finished square is then:

A D C B E
C E B A D
B C D E A
E B A D C
D A E C B

Example 4: Random choice of one or a series of 6x6 Latin squares, assuming that the letters A B C D E F have already been assigned to the variables.

9150
7604
4857
1479
8307
5690
2340
9441

1. We select at random one of the 9408 possibilities, using the random numbers to the left to determine choice. Since there are 9408 reduced squares we must use this number as our basis for a random selection. (See key numbers above each of the 22 transformation sets.) Our random choice is 9150. This means that we use the square representing numbers 9061-9180, from the list on page 11:

A B C D E F
B C A E F D
C A B F D E
D E F B A C
E F D A C B
F D E C B A

2. Selecting at random the order in which these six rows shall appear in the new square, we continue by using the last two digits of the number 7604 etc. We divide by 6, obtaining 4 3 1 ~~6~~ 6 ~~4~~ 5, and the last must be 2. The revised order of rows is 4 3 1 6 5 2, and the square is:

C A B F D E
F D E C B A
B C A E F D
A B C D E F
E F D A C B
D E F B A C

5 6 1 2 3 4

6935
3729
0984
7537
3501
0892
7551

3. Selecting at random the order in which the above columns shall appear, we divide the last two digits of the random numbers on the left successively by 6; this order is obtained: 5 ∤ 6 1 ∤ 2 3, and 4. The resulting square is:

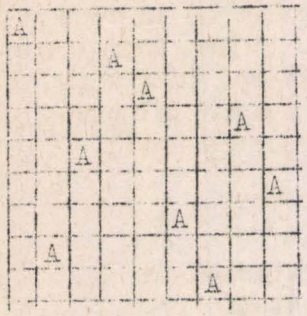
B F D E C A
E C B A F D
A E F D B C
C D E F A B
D A C B E F
F B A C D E

4641
1138
8903
5288
2850
9735
6166
3884
9002
6478
3377
0655

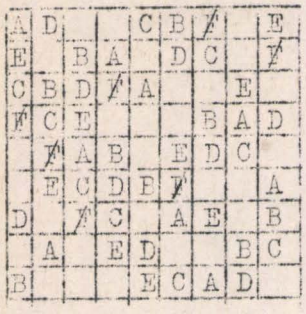
4. Selecting at random the arrangement of letters, we determine first which letter in the square above will be represented by treatment A; if numerical orders 1 2 3 4 5 6 are assigned to letters A B C D E F as shown above, reference to the last two digits of the column of random numbers on the left shows that A will replace the fifth letter or E; B will replace the second letter, B; C will replace C, D will replace D, E will replace F, and F must replace A. The resulting and finished 6x6 square when those letters are substituted, is:

B E D A C F
A C B F E D
F A E D B C
C D A E F B
D F C B A E
E B F C D A

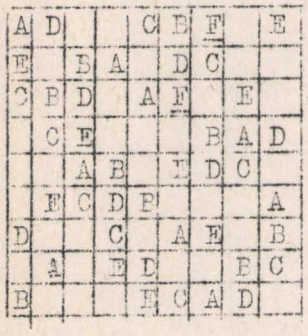
Example 5: Random choice of one of a series of 9x9 Latin squares assuming that the letters A, B, C, D, E, F, G, H, I have already been assigned to variables.



1. A is the first letter to be assigned. There are nine possible squares for A in row 1. Selection from a table of random numbers gives the choice to the first square. Eight possible squares remain for A's position in row 2. Choice is the third of the eight etc.



2. Letters B, C, D, and E are filled in similarly. Difficulties appear when F is attempted. No spaces remain in rows 8 and 9 for this letter. So, F is removed and a new system is begun.



3. We record at the right of the square a number after each row indicating the number of possible spaces for F in that row. All rows have four possibilities in the beginning. We choose at random one of the four in row 1, which happens to be the third, record the F, and change the list of possible spaces at right; rows 3, 6, and 8 now have only three possible openings for F. We choose at random one of the three spaces in row 3, which happens to be the second and change the number of possibilities listed.

It is apparent now, when the F has been located in rows 1 and 3, that some rows still have four possible spaces for their F, one has three, and two have only two spaces. We fill in first the row with least openings and in case of a tie we select the topmost one. The least possibilities for this letter are found in rows 6 and 8, for which only two of the four actual spaces can hold an F; so, in row 6 a random choice of the two spaces is next made. We repeat the above method until all F's are filled in.

A D H G C B F I E
E F B A I D C H G
C B D H A F G E I
F C E I G H B A D
I G A B H E D C F
H E C D B G I F A
D H I C F A E G B
G A F E D I H B C
B I G F E C A D H

4. The same procedure is followed for G, and H and I are quickly located. The complete square is on the left. (Caution should be exercised in using squares of higher orders, for the increased heterogeneity in a large area affects results unpredictably.)

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